NEUMANN ARRANGEMENT WAS NAMED AFTER THE MATHEMATICIAN CARL NEUMANN

Muhammad Raheel
UNIVERSITY OF OKARA

Abstract:

Questions
Q1. Write the detailed note on Neumann series. (15)
Q2. Write the detailed note on Boundary Value Problem. (15)
Q3. What are the types of partial differential equations? (10)

Q1. Write the detailed note on Neumann series.
Neumann series:
Introduction:
Neumann series was named after the mathematician Carl Neumann who used it in 1877 in the context of potential theory. The Neumann series is used in functional analysis. Most of the integral equations cannot be solved by the specialized techniques we had to develop a rather general technique for solving such integral equations. The method is applicable whenever the series converges. We can solve a linear integral equation of the second kind by making successive approximations which we can discuss later.
Mathematical Representation:
A Neumann series in mathematical form can be written as:

$$\sum_{k=0}^{\infty} T^k$$

Where T is an operator and $T^k = T^{k-1} \cdot T$, Its K times repeated application.
Uses:
Neumann series is used in functional analysis. It forms the basis of the liouville- Neumann series, which is used to solve integral equations. It is also important when studying spectrum of bounded operator.
Properties:
Let T is a bounded linear operator on the normed vector space X. If the Neumann series converges in the operator norm then (Id –T) is invertible and its inverse can results a series which can be written as:

$$(Id - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

Where; Id is an identity operator in X. Consider the partial sums:

$S_n = \sum_{k=0}^{n} T^k$, Then we can have:

$$\lim_{n\to\infty}(Id - T) S_n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1}\right) = \lim_{n\to\infty}(Id - T^{n+1}) = Id$$

- The result on operator is analogous to geometric series in R, in which we find that: (1-x) •

$(1 + x + x^2 + \dots + x^{n-1} + x^n) = 1 - x^{n+1}$

$1 + x + x^2 + \dots = \frac{1}{1-x}$

There is a one case in which convergence is guaranteed is when X is a Banach space and |T| < 1 in the operator norm or $\sum |T^n|$ is convergent.

Other Mathematical Representation:

A Neumann series in mathematical form can also be written as: $Y_{(x)} = f_{(x)} + \lambda \int^{x} k(x, t)y(t)\,dt$

Important approximations:

Zero – Order Approximation:

$$Y_0(x) = f(x)$$

First– Order Approximation:

$$Y_{(1)}(x) = f(x) + \lambda \int_a^x (x, t)_0 (t)\,dt$$

Second– Order Approximation:

$$Y_{(2)}(x) = f(x) + \lambda \int_a^x (x, t)_1 (t)\,dt$$

Put the value of $Y_{(1)}$

$$Y_{(2)}(x) = f(x) + \lambda \int_a^x (x, t)[\, f(x) + \lambda \int_a^x k(x, t)y_0(t)\,dt]$$

$$Y_{(2)}(x) = f(x) + \lambda \int_a^x (x, t)\, \lambda^2 \int_a [\ \int_a^x k(x, t)\, k(x, t)y(t)\,dt]$$

a 0

Y $_{(2)}$ (x) = f (x) + $\lambda \int^{x}$ (x, t) (t) + λ^2
∵ Y_o(x) = f (x) a
∵ (t) = f (t) Where
$x\int_a$ [$x\int_a k_2$
$(x, t)f(t)\, dt$]
Y $_n$ (x) = f (x) + $\lambda \int^{x}$ (x, t) (t) + λ^2 a
$x\int_a$ [

$x\int_a k_n$ $(x, t)f(t)\, dt$]

Example:
In order to illustrate the Neumann method, we consider the integral equation
φ(x) = x + 1 ∫1 (t − x)φ(t)dt (1) ‒
solution:
2 −1
To start the Neumann series we take the assumption such as:
φ_o(x) = x Then

1 1
φ_1(x) = x + ∫ (t − x)tdt 2
−1
φ (x) =

1
x + (t
2 3
φ (x) x + 1 1 = ‒
3

Substituting the value of φ_1(x) in equation (1) we get
φ (x) = x + 1 ∫1 (t − x)tdt + x + 1 ∫1 (t − x) 1 dt

2 2 −1 2 −1 3
φ (x) = x + 1 − x

2 3 3

Continuing this process of substituting back into equation (1) we get
φ (x) = x + 1 − x − 1

3 3 3 3^2

And by mathematical induction we get
n n

$$\varphi_{2n}(x) = x + \sum_{s=1}(-1)^{s-1}\, 3^{-s} - x \sum_{s=1}(-1)^{s-1}\, 3^{-s}$$

Letting $n \to \infty$ we get

$$\varphi(x) = \frac{3}{4}x + \frac{1}{4}$$

the solution can be verified by substituting in equation 1.

Q2. Write the detailed note on Boundary Value Problem.

Boundary value problems introduction:

Basic 2nd order Boundary value problem:

A 2nd order boundary-value problem consists of a 2nd order differential equation along with constraints on the solution y = y(x) at two values of x.

For example:

$y'' + y = 0$; With $y(0) = 0$ and $y\left(\frac{\pi}{6}\right) = 4$

This is fairly a simple boundary value problem; so we can have a problem such as:

$y'' + y = 0$; With $y'(0) = 0$ and $y'\left(\frac{\pi}{6}\right) = 4$

Or: alternatively we might not be requiring any particular values at the two points just that they are related in some way for example:

$y'' + y = 0$; With $y(0) = y\left(\frac{\pi}{6}\right)$ and $y'(0) = y'\left(\frac{\pi}{6}\right)$

The constraints given/imposed at the two points are called either boundary values or boundary conditions. Mainly the interval of interest for the differential equation is the interval between the two points at which boundary conditions are specified and these two points are often referred to as boundary points.

Or

A differential equation of order two or greater in which the dependent variable y or its derivatives are defined at different points such as:

$$g(x) = a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_o y$$

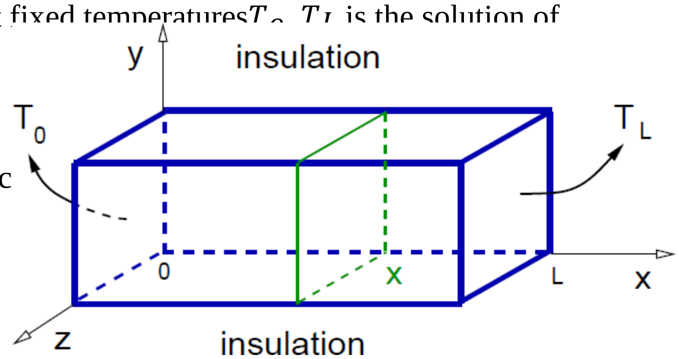
And; $y(a) = y_o$, $y(b) = y_1$, is called a two point boundary value problem or boundary value problem. The specified value $y(a) = y_o$, $y(b) = y_1$ are called boundary conditions.

Solution to a given boundary-value problem:

Solution to a given boundary-value problem is a function that satisfies the given differential equation over the specified interval or interval of interest along with the given boundary conditions.

For example:
Let us solve the boundary-value problems involving the differential equation
$y'' + y = 0$
Having the general solution as:
$Y(x) = c_1 \cos(x) + c_2 \sin(x)$
The only work in solving such boundary-value problem is to determine the values of c_1 and c_2 by the given boundaries values.
Classes of Boundary Conditions:
There are only three classes of boundary conditions that arise in practice. They are the Regular boundary conditions, Boundedness boundary conditions and Periodic boundary conditions.
Regular boundary conditions:
A boundary condition at $x = x_o$ is said to be regular if and only if it can be described by the relation:
$\alpha y(x_o) + \beta y'(x_o) = \gamma$
Whereα, β and γ are constants and α or β or both being nonzero.
If either α or β is often zero the equations reduced to For $\alpha = 0$; $y'(x_o) = \gamma$
For β=0; $(x_o) = \gamma$
And in many cases $\gamma = 0$
Boundedness boundary conditions:
When a solution does not "blow up" at a point $x = x_o$. To be precise we can write:
$\lim_{x \to x_o} |y(x)| < \infty$
We usually write this as;$|y(x)| < \infty$ such conditions are typically the appropriate conditions when x_oa singular point for the differential equation is.

Periodic boundary conditions:
Periodic boundary condition states that the solution or its derivatives at two distinct points $x = x_o$ and $x = x_1$ are equal i.e.
$y(x_o) = y(x_1)$ or $y'(x_o) = y'(x_1)$
These two periodic boundary conditions occur together and these conditions naturally occur when the variable x is actually the angular component θ in a polar coordinate system. The first two types of boundary conditions, regular and boundedness are often said to be separated boundary conditions, since they can be imposed separately at each boundary point.
Homogeneous Boundary-Value Problems:
2nd order differential equation
$ay'' + by' + cy = g,$
Is said to be homogenous if and only if g= 0.
The solutions to these equations satisfied the "principle of superposition".
Example from physics:
Problem: The equilibrium (time independent) temperature of a bar of length L with insulated horizontal sides and the bar vertical extremes kept at fixed temperatures T_0, T_L is the solution of the BVP:
$T''(x) = 0,\ x \in (0, L),\ T(0) = T_o,\ T(L) = T_L$



Types of boundary conditions:
There are three kinds of boundary conditions which c
Dirichlet boundary conditions.
Neumann boundary conditions.
Cauchy boundary conditions.

Dirichlet boundary conditions:
Dirichlet boundary conditions are known as 1st type boundary condition in which the value of function f is specified / defined at the boundary.
Neumann boundary conditions:
Neumann boundary conditions are known as 2nd type boundary condition in which normal derivative of the function f i.e. $\frac{\partial f}{\partial n}$ is specified on the boundary.
Cauchy boundary conditions:
Cauchy boundary conditions are known as mixed boundary conditions because a function f and its normal derivative $\frac{\partial f}{\partial n}$ is specified on the boundary which is in the form of a curve or a surface that gives a value to the normal derivative and the variable or a function f itself.

Q3. What are the types of partial differential equations? Differential Equation: Introduction:
An equation involving one dependent variable and its derivatives w.r.t one or more independent variable is called Differential equation. For example

$$\frac{dy}{dx} + y\cos x = \sin x.$$

Ordinary Differential Equation:
A differential equation in which ordinary derivatives of the dependent variable w.r.t a single independent variable occur is called an Ordinary Differential Equation. For example

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$$

Partial Differential equation:
A differential equation involving partial derivatives of the dependent variables w.r.t more than one independent variable is called Partial Differential equation. For example

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Types of Partial Differential equation:

First order PDE's:

A 1st order partial differential equation with n independent variables has the general form

$$F\left(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right) = 0$$

Where; $u = u(x_1, x_2, \dots, x_n)$ is the unknown function.

1st order quasilinear partial differential equation

~~A 1st order quasilinear partial differential equation with two~~ independent variables has the general form

$$f(x, y, u)\frac{\partial u}{\partial x} + g(x, y, u)\frac{\partial u}{\partial y} = h(x, y, u)$$

Such equations are encountered in various applications such as

Gas dynamics.

Hydrodynamics.

Heat and mass transfer.

Wave theory.

First order non-linear PDE's:

The partial differential equations which cannot be represented in term of linear equation are called First order non-linear PDE's.

Homogeneous linear partial differential equation:

If all the partial derivatives involved in the differential equation are of the same order then the equation is called Homogeneous linear partial differential equation.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} = 0$$; is homogenous.

Non-Homogeneous linear partial differential equation:

If all the partial derivatives involved in the differential equation are not of the same order then equation is called non-Homogeneous linear partial differential equation.

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3} = x_1 + x_2$$

is non-homogenous.

Second order PDE's:

A 2nd order linear partial differential equation with two independent variables has the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} = 0$$

Types of 2nd order PDE'S

~~There are three-types of 2nd~~ order PDEs in mechanics which are classified as:

Elliptic PDE'S

Hyperbolic PDE'S.

Parabolic PDE'S.

Elliptic PDEs:

The equations of elasticity without inertial terms are known as elliptic PDEs. The solutions of elliptic PDEs are always smooth even if the initial and boundary conditions are rough (though there may be singularities at sharp corners). Boundary data at any point may affect the solution at all points in the domain. The PDE is called elliptic if $b^2 - 4ac < 0$, an example is

Hyperbolic PDE'S:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_1\,\partial x_2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_1} = x$$

Hyperbolic PDEs describe wave propagation phenomena. In hyperbolic PDEs the smoothness of the solution depends on the smoothness of the initial and boundary conditions. If there is a jump or discontinuity in the data at the start or at the boundaries then the jump will propagate as a shock in the solution. If the PDE is nonlinear then shocks may develop even though the initial conditions and the boundary conditions are smooth. In a system modeled with a hyperbolic PDE information travels at a finite speed referred to as the wave speed. Information is not transmitted until the wave arrives. The PDE is called hyperbolic if $b^2 - 4ac > 0$; an example is

Parabolic PDE'S:

$$\frac{\partial^2 u}{\partial x_1^2} + 3\frac{\partial^2 u}{\partial x_1\,\partial x_2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_1} = x$$

The heat conduction equation is an example of a parabolic PDE's. Parabolic PDEs are time dependent and represent diffusion-like processes the solutions obtained are smooth in space but may possess singularities. The information can travels at infinite speed in a parabolic system. The PDE is called parabolic if $b^2 - 4ac = 0$; an example is

$$\frac{\partial^2 u}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_1\,\partial x_2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_1} = x$$

Importance in Physics:

Laplace's equation:

Many time-independent problems are describes by Laplace's equation which is defined for $\varphi = \varphi(x, y, z)$, $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$

Some examples of Laplace's equation are the electrostatic potential in a charge or free region the gravitational potential in a matter or gravitational potential in free region, the steady-state temperature in a region with no heat source.

Poisson's equation:

Equation is like a Laplace's equation except that it allows an inhomogeneous term $f(x, y, z)$ known as the source density & has the form

$\nabla^2\varphi = f(x, y, z)$.

Heat equation:

Wave propagation including waves on strings, pressure waves in gasses, liquids or solids, electromagnetic waves and gravitational waves, and the current or potential along a transmission line all satisfy the wave equation

Schrodinger equation: $-\frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} + \nabla^2\varphi = 0$

The Schrodinger equation will be

$$-\frac{\hbar^2}{2m}\nabla^2\varphi + V(x, y, z)\varphi = i\hbar\frac{\partial\varphi}{\partial t},$$

This equation explains the time dependence of the wave function of a particle moving in a given potential $V(x, y, z)$.